

The epidemic threshold in directed networks

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Abstract

Epidemics have so far been mostly studied in undirected networks. However, many real-world networks, such as the social network Twitter and the WWW networks, upon which information, emotion or malware spreads, are shown to be directed networks, composed of both unidirectional links and bidirectional links. We define the *directionality* ξ as the percentage of unidirectional links. The epidemic threshold τ_c for the susceptible-infected-susceptible (SIS) epidemic has been proved to be $1/\lambda_1$ in directed networks by N-intertwined Mean-field Approximation, where λ_1 , also called as spectral radius, is the largest eigenvalue of the adjacency matrix. Here, we propose two algorithms to generate directed networks with a given degree distribution, where the directionality ξ can be controlled. The effect of ξ on the spectral radius λ_1 , principal eigenvector x_1 , spectral gap $(\lambda_1 - |\lambda_2|)$ and algebraic connectivity $|\mu_{N-1}|$ is studied. Important findings are that the spectral radius λ_1 decreases with the directionality ξ , and the spectral gap and the algebraic connectivity increase with the directionality ξ . The extent of the decrease of the spectral radius depends on both the degree distribution and the degree-degree correlation ρ_D . Hence, the epidemic threshold of directed networks is larger than that of undirected networks, and a random walk converges to its steady-state faster in directed networks than in undirected networks with degree distribution.

1 Introduction

Much effort has been devoted to understand epidemics on networks, mainly because of the increasing threats from cybercrime and the expected outbreak of a new fatal virus in populations. Epidemics have been studied on undirected networks for a long time and many authors (see [1, 2, 3, 4, 5, 6, 7]) addressed the existence of an epidemic threshold τ_c in the susceptible-infected-susceptible (SIS) epidemic process [8]. We consider the SIS epidemic process in an undirected network $G(N, L)$, characterized by a symmetric adjacency matrix A consisting of elements a_{ij} that are either one or zero depending on whether node i is connected to j or not. Each node i has two possible states at time t : either $X_i(t) = 0$ meaning that the node is healthy and susceptible or $X_i(t) = 1$ for an infected node. Initially, a certain percentage of nodes is randomly selected as infected. The infection from an infected node to each of its healthy neighbors and the curing of an infected node are assumed to be independent Poisson processes with rates β and δ , respectively. Every node i at time t is either infected, with probability $v_i(t) = \text{Prob}[X_i(t) = 1]$ or healthy (but susceptible) with probability $1 - v_i(t)$. This is the general continuous-time description of the simplest type of a SIS epidemic process on a network. The epidemic threshold τ_c separates two different phases of the epidemic process on a network: if the effective infection rate $\tau \triangleq \beta/\delta$ is above the threshold, the infection spreads and becomes persistent; if $\tau < \tau_c$, the infection dies out exponentially fast. The epidemic threshold $\tau_c^{(1)} = \frac{1}{\lambda_1}$ of the SIS model by the N-intertwined Mean-field Approximation (NIMFA) [7, 9, 10] lower bounds the real threshold, where λ_1 is the largest eigenvalue of the adjacency matrix A , also called the spectral radius.

Table 1: The percentage of unidirectional links in real-world networks

Real-world networks	N	L	ξ
Enron	69 244	276 143	84.29%
Ljournal-2008	5 363 260	79 023 142	25.32%
Twitter-2010	41 652 230	1 468 365 182	64.29%
WordAssociation-2011	10 617	72 172	76.77%
cnr-2000	325557	3216152	70.33%
in-2004	1382908	16917053	60.68%
eu-2005	862664	1935140	67.80%
uk-2007-05@100000	100000	3050615	82.23%
uk-2007-05@1000000	1000000	41247159	79.71%

Topologies of complex networks have been mostly modeled by Erdős and Rényi¹ [11] as binomial networks or by Bárábasi and Albert² [12] as power-law networks or by Watts and Strogatz³ [13] as small-world networks. However, many real-world networks are *directed* networks as shown in Table 1. Two kinds of links, namely bidirectional links and unidirectional links, exist in directed networks. If node i is connected to node j then j is also linked to i , one bidirectional link exists between nodes i and j ; and if either $i \rightarrow j$ or $j \rightarrow i$ exists, but not both in between a node pair i and j , a unidirectional link exists. Here, we define the directionality as $\xi = L_{\text{unidirectional}}/L$, where the number of links $L = \sum_i \sum_j a_{ij} = u^T A u$ and $L = L_{\text{unidirectional}} + 2L_{\text{bidirectional}}$, because each bidirectional link consists of two unidirectional links with opposite directions. A directed network with the directionality ξ is denoted by $G^{(\xi)}$. When $\xi = 0$, the network $G^{(\xi=0)}$ is a purely bidirectional network, whose adjacency matrix is symmetric, *i.e.* $G^{(\xi=0)}$ and its corresponding undirected network has the same spectral properties. When $\xi = 1$, the network $G^{(\xi=1)}$ is a directed network without any bidirectional link. A high directionality is observed in Twitter, as shown in Table 1. A link runs from user A to user B if user A follows user B in Twitter, where user A is called the "follower" of user B. Just because user A is user B's "follower" does not necessarily mean that the reverse is also true. For example, a famous football player could have millions of followers but he may not follow many others. This explains the high directionality ξ of Twitter. Another type of social networks, the virtual-community social networks, such as Facebook, Chinese Renren and LiveJournal, have low directionality. One of the reasons for building this kind of websites is to keep connection with real-life friends and classmates. If two people know each other in real-life, they have a high chance to be friends with each other, which means that there are more bidirectional links in this kind of networks. In other words, the directionality of this type of social networks is low, which also can be observed in Table 1. The data set of the real-world networks is obtained from [14, 15, 16] and the description of these networks is attached in the Appendix A. Garlaschelli and Loffredo [17] researched the reciprocity in directed networks, where the reciprocity is equal to $1 - \xi$. Here, we focus on how the directionality ξ influences the epidemic threshold and other spectral properties.

Stimulated by the directed social networks, we study the SIS epidemic threshold in directed networks. Percolation theory for directed networks with $\xi = 1$ was firstly developed by Newman and co-workers [18, 19]. Boguñá and Serrano [20] pointed out that even a small fraction of bidirectional links suffices to percolate the network. Moreover, the bidirectional links will speed up some propagation process [17, 21]. The epidemic

¹An Erdős-Rényi random graph can be generated from a set of N nodes by randomly assigning a link with probability p to each pair of nodes.

²A Bárábasi-Albert graph starts with m nodes. At every time step, we add a new node with m links that connect the new node to m different nodes already present in the graph. The probability that a new node will be connected to node i in step t is proportional to the degree $d_i(t)$ of that node. This is referred to as preferential attachment.

³A Watts-Strogatz small-world graph can be generated from a ring lattice with N nodes and k edges per node, by rewiring each link at random with probability p .

threshold $\tau_c^{(1)} = \frac{1}{\lambda_1}$ of the SIS epidemic process by NIMFA also holds for directed networks in [22]. This motivates us to investigate the influence of the directionality ξ on the epidemic threshold $\tau_c^{(1)} = \frac{1}{\lambda_1}$ in directed networks. Recently, Chen and Olvera-Cravioto [23] proposed an algorithm to generate a directed network with given a degree distribution, which is similar to the configuration model. However, Chen and Olvera-Cravioto's algorithm in [23] cannot generate a directed network with a precise directionality ξ . Here we propose two algorithms to generate directed networks with a given degree distribution and a given directionality ξ , where ξ is a *control parameter*.

Topological properties of directed networks, such as the short loops, closure connectivity, degree distributions, domination and communities on realistic directed networks have been studied in [24, 25, 26, 27]. Dynamic processes on real directed networks have also been researched in [28, 29]. Here, we investigate the effect of the directionality ξ on the spectral radius λ_1 , the principal eigenvector x_1 , the spectral gap ($\lambda_1 - \lambda_2$) and the algebraic connectivity μ_{N-1} in both binomial and power-law directed networks. We call a network a binomial⁴ (or power-law) directed network, if its in-degree and out-degree follow the same binomial (or power-law) distribution. Furthermore, we explore the influence of the linear degree correlation ρ_D (also called the assortativity) on the epidemic threshold $\tau_c^{(1)}$ besides the directionality ξ in both binomial and power-law directed networks.

Interestingly, the spectral radius λ_1 of networks $G^{(\xi=0)}$ is always larger than that of directed networks $G^{(\xi=1)}$ when the degree distribution and the assortativity of these networks are the same. It means that the epidemic threshold τ_c in undirected networks is smaller than that in directed networks with the same degree distributed and assortativity. Moreover, the decrease of the spectral radius λ_1 with ξ is large when the assortativity is large in binomial directed networks, whereas the opposite is observed in power-law directed networks. We conclude that a raising of the percentage of unidirectional links increases the epidemic threshold and unidirectional links impose an addition constraint on the spread of information/viruses.

2 Algorithm Description

The networks mentioned in this paper are simple, without self-loops and multiple links in the same direction. One of the main goals in this work is to propose several algorithms for generating a directed network with a given directionality ξ . The in-degree and out-degree preserving rewiring algorithm proposed in this section, is inspired by the degree preserving rewiring, which is presented in [30]. Here, we firstly introduce the degree-preserving rewiring in undirected networks, that monotonously increases or decreases the linear degree correlation ρ_D (also called the assortativity), while maintaining the node degrees unchanged. Then, we describe the in-degree and out-degree preserving rewiring algorithm and the link resetting algorithm in detail.

2.1 Degree-preserving rewiring

The degree-preserving rewiring [30] can either increase or decrease the assortativity of a network: (a) the degree-preserving assortative random rewiring: randomly select two links associated with four nodes and then rewire the two links such that the two nodes with the highest degree and the two lowest-degree nodes are connected. If any of the new links exists before rewiring, discard this step and a new pair of links is randomly selected; (b) the degree-preserving disassortative random rewiring: randomly select two links associated with four nodes and then rewire the two links such that the highest-degree node and the lowest-degree node are connected, while also the remaining two nodes are connected provided the new links do not exist before. Either rewiring step (a) or (b) can be repeated to monotonically increase or decrease the assortativity.

⁴For example, an Erdős-Rényi random network is a binomial network with the linear degree correlation (also called the assortativity) $\rho_D = 0$. A general binomial network could possibly have an assortativity within a large range.

2.2 In-degree and out-degree preserving rewiring algorithm (IOPRA)

We define our in-degree and out-degree preserving rewiring algorithm (termed IOPRA) as follows: randomly choose two unidirectional links with four nodes, and rewire the two unidirectional links. In IOPRA, the head of one unidirectional link only can rewire with the head of another unidirectional link, in order to maintain both the in-degree and out-degree of the four nodes unchanged (see Figure 1). If any link exists between the two new node pairs before rewiring, or if the directionality of the new network does not increase (or decrease) towards the given directionality, discard this step and randomly select another unidirectional link pairs. We illustrate the process of IOPRA in Algorithm 1 (see Appendix C). IOPRA actually changes directionality ξ of a given network G without changing the in and out-degree of each node. It means that the degree distribution of the directed network $G^{(\xi)}$ comes from the original network G . If the original network is an undirected network, the in-degree sequence is exactly the same as the out-degree sequence in the directed network.

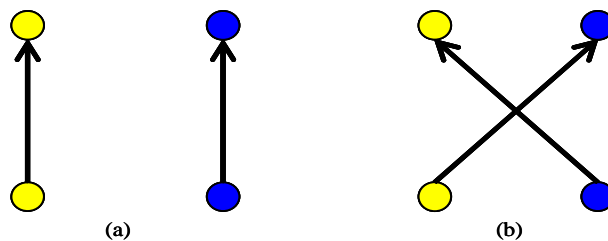


Figure 1: (Color online) The in-degree and out-degree preserving rewiring

Applying the algorithms IOPRA to an initial network actually randomizes the connections of the original network to change the directionality without changing the degree of each node. Hence, if the initial network is e.g. an ER random network or a BA power-law network, where the connections are originally random, IOPRA changes only the directionality ξ . However, if we apply IOPRA to e.g. a lattice, the resulting network has not only a different directionality but a more randomized structure.

2.3 Link resetting algorithm (LRA)

LRA can only generate random directed networks with a binomial degree distribution. We start with an Erdős and Rényi network $G_p(N)$ with N and p , and use LRA to change the directionality. We randomly choose fraction ξ of bidirectional link pairs from $G_p(N)$. Then, we randomly choose only one unidirectional link from each bidirectional link, and relocate the unidirectional links to the places without any link. A directed network $G^{(\xi)}$ with directionality ξ is obtained (see Algorithm 2 in Appendix C). LRA is used to generate binomial directed networks with controllable directionality ξ . As the links are randomly reallocated, the in-degree and out-degree of every node in the new network is not exactly the same as that of the original bidirectional network $G_p(N)$. However, the directed network obtained by LRA still has the same binomial degree distribution for both in and out-degree. In this work, two types of binomial directed networks can be generated: one is generated by IOPRA (called the IOPRA binomial directed networks), whose nodes have the same in-degree and out-degree; and the other created by LRA (called the LRA binomial directed networks), has different in-degree and out-degree sequences, which is more general in real-world networks.

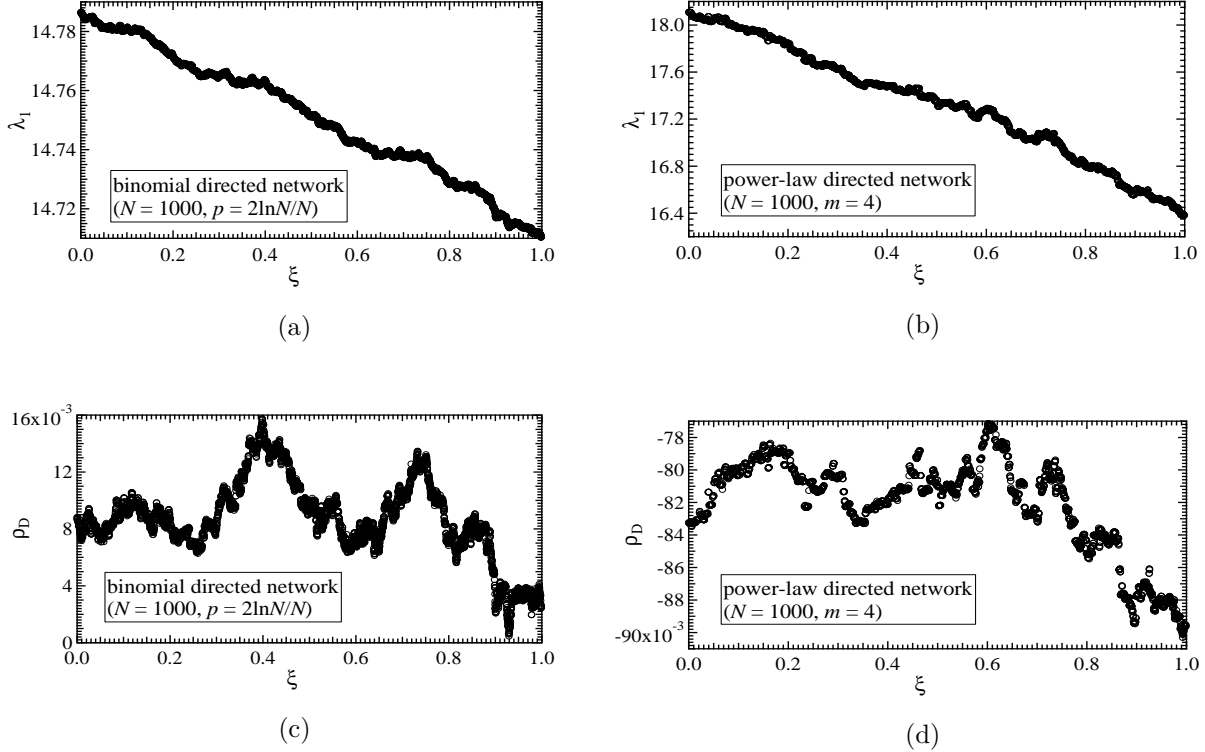


Figure 2: Plot of the spectral radius versus the directionality in (a) and (b), as well as the assortativity versus the directionality in (c) and (d), in both binomial and power-law directed networks generated by IOPRA.

3 Spectral properties in directed networks

3.1 Spectral radius of the directed networks

The adjacency matrix of the directed network is an asymmetric matrix, whose spectral radius λ_1 is still real by the Perron–Frobenius Theorem (see [31]). We generate the directed networks from the ER ($N = 1000$, $p = 2\ln N/N$) and BA ($N = 1000$, $m = 4$) networks gradually using IOPRA, with the directionality ξ ranging from 0 to 1. The influence of the directionality ξ on the spectral radius λ_1 and the assortativity ρ_D is studied in both power-law directed networks and binomial directed networks (see Figure 2).

Apart from some wobbles, the spectral radius λ_1 almost linearly decreases with the directionality ξ . Moreover, the assortativity ρ_D of the network fluctuates slightly around 0. We observe that the tiny leaps of spectral radius λ_1 happen when the assortativity ρ_D has a rise, which is understandable, because it has been presented in [30] that the spectral radius λ_1 increases with the assortativity ρ_D . Figure 3 examples that the spectral radius λ_1 may increase instead of decrease when the directionality increases due to the assortativity ρ_D . We will study the effect of the assortativity ρ_D on the decrease of the spectral radius λ_1 with the directionality ξ in Section 4 later.

With LRA, we generate binomial directed networks with directionality ξ from 0 to 1 with step 0.1. The assortativity ρ_D of all the binomial directed networks generated by LRA is around 0. The effect of the assortativity ρ_D can be ignored here. The spectral radius λ_1 is calculated in directed networks with different directionality ξ . We performed all the simulations for 10^3 network realizations. The spectral radius λ_1 is plotted as a function of the directionality ξ in binomial directed networks with $p = 2\ln N/N$ and $p = 0.05$ in Figure 4. From the observation, the spectral radius λ_1 decreases linearly with the directionality ξ with the slop around -1 , which is independent from the link density p of the networks. We make a proposition which is used to calculate the spectral radius λ_1 of the binomial directed networks with directionality ξ .

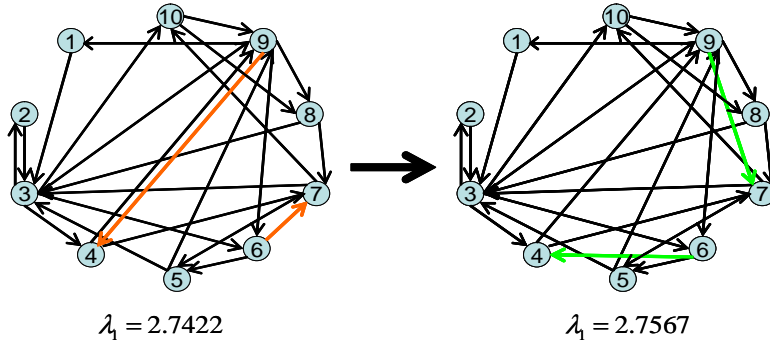


Figure 3: (Color online) Example: the spectral radius increases with the directionality ξ , because of the increase of the assortativity (where $\rho_D(G_{left}) = -0.6190$, $\xi(G_{left}) = 0.8333$, and $\rho_D(G_{right}) = -0.5714$, $\xi(G_{right}) = 0.9167$).

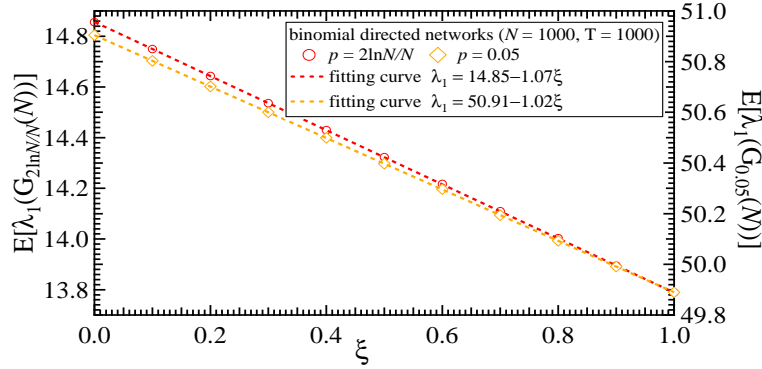


Figure 4: (Color online) The average of the spectral radius as a function of the directionality for binomial directed networks generated by LRA with size $N = 1000$. Two values for the link density p are shown: $p = 2\ln N/N$ (circles) and $p = 0.05$ (diamonds).

Proposition 1 Let $G_p(N)$ be an Erdős-Rényi (ER) random network with N nodes and link density p , and $G^{(\xi)}$ be a binomial directed network generated by LRA whose in and out-degree follow the same binomial distribution as $G_p(N)$. The average of the spectral radius of these two types of networks satisfies

$$E[\lambda_1(G^{(\xi)})] \simeq E[\lambda_1(G_p(N))] - \xi \quad (1)$$

Arguments:

To analyse the spectral radius λ_1 of a binomial directed network generated by LRA with link density p , we consider that the network is obtained by adding $2p\xi\binom{N}{2}$ unidirectional links to a bidirectional ER network with size N and link density $p(1 - \xi)$. The average spectral radius of ER networks with N and p is $E[\lambda_1(G_p(N))] = (N - 2)p + 1 + O(\frac{1}{\sqrt{N}})$. Thus, the average spectral radius of ER network with size N and link density $p(1 - \xi)$ is

$$E[\lambda_1(G_{p(1-\xi)}(N))] = (N - 2)p(1 - \xi) + 1 + O(\frac{1}{\sqrt{N}}).$$

The principal eigenvector of an adjacency matrix A is denoted by x_1 obeying the normalization $x_1^T x_1 = 1$. Let C denote the adjacency matrix of the resulting network after adding one unidirectional link to network G . If one element in a non-negative matrix A is increased, then the largest eigenvalue also increased [31, pp. 236, Lemma 7] as

$$\lambda_1(C) \geq \lambda_1(A) + (x_1)_i (x_1)_j.$$

Hence, the average increase of the spectral radius by adding m unidirectional links in random networks is obtained as

$$E[\lambda_1(C) - \lambda_1(A)] \geq mE[(x_1)_i (x_1)_j].$$

The sum of the product of components in the principal eigenvector of Erdős and Rényi networks is approximated by a function of link density p (see Figure 5). When $N \rightarrow \infty$, and $p \neq 1$ or 0, the fitting function can be expressed as,

$$E \left[\sum_{j=1}^N \sum_{i=1}^N (x_1)_i (x_1)_j \right] = N - \frac{1}{p} \pm O(1).$$

Since $\sum_{i=1}^N (x_1)_i^2 = 1$ and using the linearity of the Expectation Operator $E[X]$, we obtain

$$E[(x_1)_i (x_1)_j] \approx \frac{N - \frac{1}{p} - 2}{N(N-1)}. \quad (2)$$

Hence, the average of spectral radius of the directed network by adding $m = 2p\xi\binom{N}{2}$ unidirectional links to network $G_{p(1-\xi)}(N)$ is,

$$E[\lambda_1(G^{(\xi)})] \simeq E[\lambda_1(G_{p(1-\xi)}(N))] + 2(N(N-1)/2)p\xi E[(x_1)_i (x_1)_j].$$

Using (2),

$$\begin{aligned} E[\lambda_1(G^{(\xi)})] &\simeq (N-2)p(1-\xi) + 1 + 2\left(\frac{N(N-1)}{2}\right)p\xi \frac{(N - \frac{1}{p} - 2)}{N(N-1)} \\ &= (N-2)p + 1 - \xi, \end{aligned}$$

which leads to (1). \square

Juhász [32] also pointed out that the largest eigenvalue $\lambda_1(G^{(\xi=1)})$ of a random directed network with link density p and size N is almost surely Np , when $N \rightarrow \infty$. In ER random networks, the spectral radius $E[\lambda_1(G^{(\xi=0)})] = (N-2)p + 1 + O(\frac{1}{\sqrt{N}}) \rightarrow Np + 1$, when $N \rightarrow \infty$. Both earlier results are consistent with our proposition and support that the slope of the decrease of the spectral radius λ_1 with directionality ξ is around -1 .

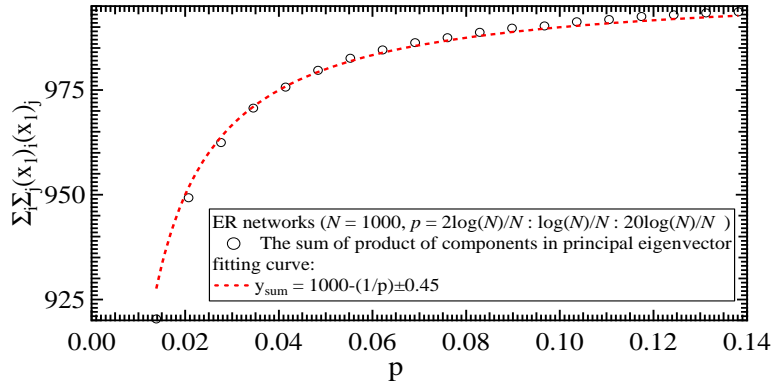


Figure 5: (Color online) Sum of the product of components in the principal eigenvector as a function of the link density p in ER networks ($N = 1000$).

With Proposition 1, we analyse the effect of the size N on the largest decrease of the spectral radius $\Lambda = \frac{\lambda_1(G^{(\xi=0)}) - \lambda_1(G^{(\xi=1)})}{\lambda_1(G^{(\xi=0)})}$. We make the prediction that $\Lambda \rightarrow 0$ if $N \rightarrow \infty$ for binomial directed networks, since the decrease of the spectral radius is almost a constant value, meanwhile, the spectral radius $\lambda_1(G^{(\xi=0)})$ of dense binomial directed network is large and increases with the size of the networks. This implies that the effect of directionality ξ on the spectral radius is small in large binomial networks.

3.2 Principal eigenvector in the directed networks

In view of the importance of the principal eigenvector x_1 in characterizing the influence of nodes on the spectral radius, in this section, we explore the principal eigenvector in directed networks. The principal eigenvector of the directed networks with the directionality ξ from 0 to 1 with step 0.1, are calculated. Then, the components in the principal eigenvector are sorted in an ascending order. The simulations for 10^3 network realizations are performed. Figure 6 illustrates that the components in the principal eigenvector are more uniform in binomial directed networks. The principal eigenvector $x_1 \rightarrow \frac{u}{\sqrt{N}}$, where u is the all-one vector, when the network has a higher directionality ξ . Moreover, the variance of components in the principal eigenvector linearly decreases with the directionality ξ in both binomial directed networks and the power-law directed networks (see Figure 7). The decrease of the variance $Var[x_1]$ in binomial directed networks by LRA is larger than that in binomial directed networks by IOPRA. Thus, nodes in LRA binomial directed networks have more equal contributions to the spectral radius than nodes in IOPRA binomial directed networks, when the directionality is the same. The difference between LRA and IOPRA binomial directed networks lies in the fact that LRA allows each node to have a different in- and out- degree. The in- and out- degree distribution are the same in both LRA and IOPRA binomial networks. Hence, the connections in LRA binomial directed networks are more random contributing to a smaller $Var[x_1]$. Li *et al.* [33] has shown that both a large variance of nodes' degree and a large assortativity ρ_D contribute to a large variance $Var[x_1]$ of the components in the principal eigenvector x_1 . Here, we point out that the directionality ξ is also related to the variance $Var[x_1]$ of the components in x_1 .

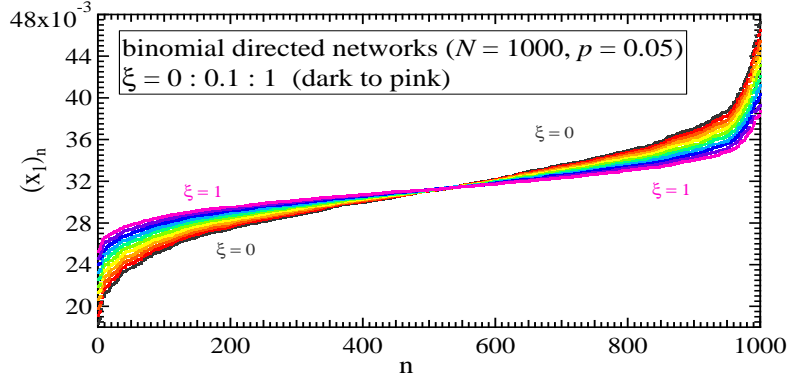


Figure 6: (Color online) The change of the components of principal eigenvector from bidirectional networks to directed networks.

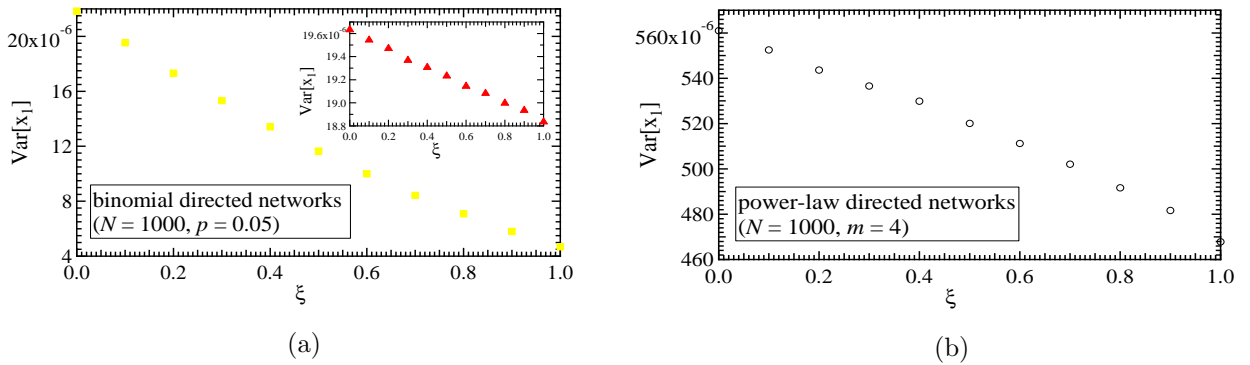


Figure 7: (Color online) Plot of the variance of the principal eigenvector versus the directionality (a) in binomial directed networks (generated by LRA in square and by IOPRA in triangle) and (b) in power-law directed networks by IOPRA (10^3 network realizations).

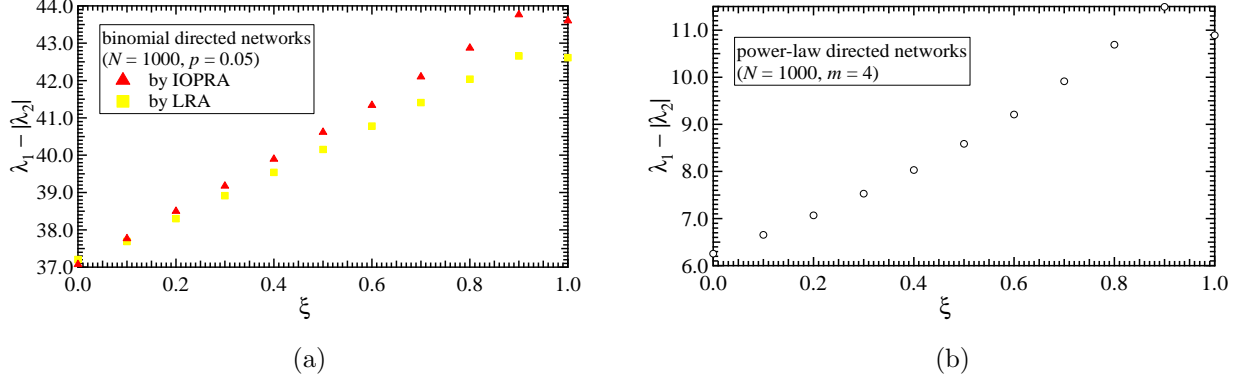


Figure 8: (Color online) Plot of the spectral gap as a function of the directionality (A) in binomial directed networks (generated by LRA in square and by IOPRA in triangle) and (B) in power-law directed networks (10^3 network realizations).

3.3 Spectral gap of the directed networks

The difference between the largest eigenvalue λ_1 and the second largest λ_2 is called the spectral gap. All eigenvalues of the symmetric adjacency matrix of an undirected network are real. Here we focus on the directed networks, whose adjacency matrix is asymmetric. The eigenvalues of the directed networks can be complex numbers. Normally, the modulus is used to measure the magnitude of an imaginary number. The spectral gap $\lambda_1 - |\lambda_2|$ increases with the directionality ξ in both the directed binomial networks and the power-law directed networks (see Figure 8). The larger the spectral gap is, the faster the random walk converges to its steady-state. Thus, the dynamic process in the directed network reaches the steady-state faster than that in the undirected network with the same degree distribution. Figure 8 (a) implies that a dynamic process is slightly faster to reach the steady-state in IOPRA binomial directed networks than in LRA binomial directed networks. The existence of large spectral gap together with a uniform degree distribution results in higher structural sturdiness and robustness against node and link failures. Hence, directed networks with high directionality ξ and a uniform degree distribution are more robust than undirected networks with large variance of degree.

3.4 Algebraic connectivity of the directed networks

The Laplacian $Q = \frac{1}{2}BB^T = \Delta - \bar{A}$, where the incidence matrix B is an $N \times L$ matrix with elements [31]

$$b_{il} = \begin{cases} 1 & \text{if link } e_l = i \rightarrow j \\ -1 & \text{if link } e_l = j \rightarrow i \\ 0 & \text{otherwise,} \end{cases}$$

$\Delta = \frac{1}{2}(\Delta_{in} + \Delta_{out})$, Δ_{in} and Δ_{out} are the in-degree and out-degree matrices, and $\bar{A} = \frac{1}{2}(A + A^T)$. If the network is an undirected network, \bar{A} is the adjacency matrix A and $\Delta = \text{diag}(d_1, d_2, \dots, d_N)$ is the degree matrix. The second smallest eigenvalue μ_{N-1} of the Laplacian Q was named by Fiedler [34]. Here we study the algebraic connectivity in directed networks, The algebraic connectivity is always mentioned together with the spectral gap, since they may both be used to quantify the robustness and the network's well-connectedness. The larger the algebraic connectivity is, the more difficult it is to cut the network into disconnected parts. Here, we study the influence of the directionality ξ on the algebraic connectivity μ_{N-1} in directed networks. The Laplacian Q of a directed network is asymmetric, thus, the algebraic connectivity μ_{N-1} might be a complex number and the modulus $|\mu_{N-1}|$ is used to measure the magnitude of the algebraic connectivity. We point out that, compared to the real part, the imaginary part of the algebraic connectivity is rather small, which can even be ignored. The algebraic connectivity increases with the directionality ξ in both the binomial directed

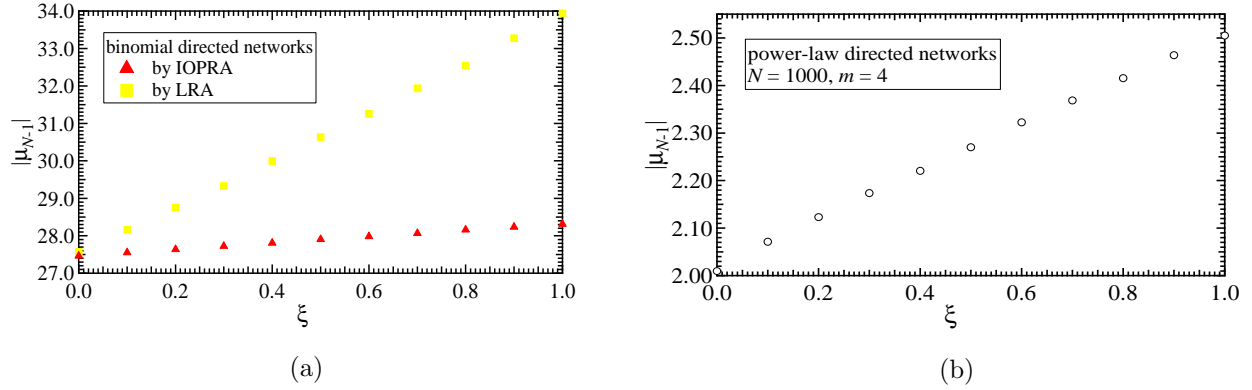


Figure 9: (Color online) Plot of the algebraic connectivity as a function of the directionality (A) in binomial directed networks (generated by LRA in square and by IOPRA in triangle) and (B) in power-law directed networks (10^3 network realizations).

networks and the power-law directed networks (see Figure 9). It illustrates that the directed networks are more difficult to break into parts. Moreover, the LRA binomial directed networks is more strongly connected than the IOPRA binomial directed networks with the same in- and out- degree distribution.

4 Effects of the assortativity on the spectral radius of the directed networks

In Section 3, we have discussed how the spectral properties change with the directionality in directed networks, where the assortativity is always around zero. Here, we study further how the spectral radius λ_1 changes with the directionality ξ when the assortativity ρ_D is the same, and how the change alters with the assortativity. Two approaches are applied to investigate this problem.

Approach 1: We firstly do degree-preserving rewiring on ER networks (or BA networks) to obtain a set of bidirectional networks with assortativity ρ_D from -0.8 to 0.8 (or -0.3 to 0.3) with step 0.1 . Secondly, alter the directionality ξ of all bidirectional networks with each assortativity using IOPRA. The directionality ξ is changed from 0 to 1 with step 0.1 . IOPRA randomizes network connections, and thus push the assortativity of the resulting directed network toward zero if the original network has a non-zero assortativity. Then, the spectral radius λ_1 of all directed networks with ξ from 0 to 1 with step 0.1 , is scatter plotted as a function of the assortativity ρ_D in different colorful lines, from the top gray line to the bottom pink line. Figure 10 plots the

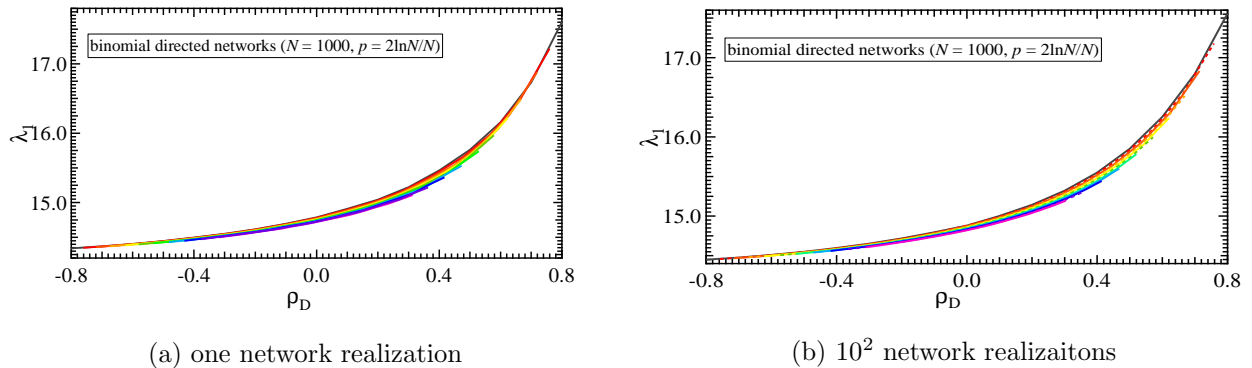


Figure 10: (Color online) The spectral radius as a function of the assortativity in binomial directed networks.

simulation result of one binomial network realization and 10^2 binomial network realizations. The simulation of one realization is almost same as the result of a large number of network realizations, which points to *almost sure behavior* [35]. The results in the power-law directed networks are shown in Figure 11.

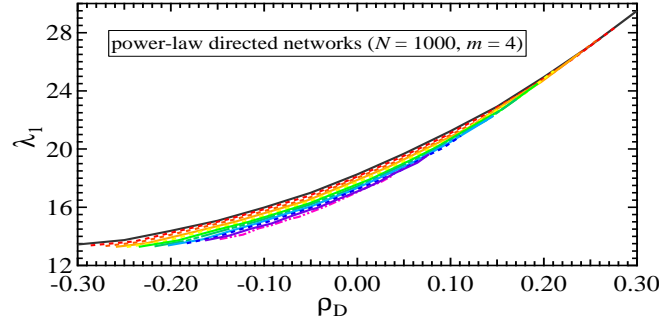


Figure 11: (Color online) The spectral radius as a function of the assortativity in power-law directed networks.

Approach 2: Firstly, generate ER networks (or BA networks) $G^{(\xi=0)}$ whose directionality $\xi = 0$, then, use IOPRA to generate binomial directed networks (or power-law directed networks) $G^{(\xi=1)}$ with directionality $\xi = 1$. Secondly, do degree-preserving rewiring on ER networks (or BA networks) $G^{(\xi=0)}$ to change the assortativity of the undirected networks, and the spectral radius of networks are calculated and plotted as a function of the assortativity (squares in Figure 12). Then, apply IOPRA to change the assortativity of the binomial directed networks (or power-law directed networks) $G^{(\xi=1)}$. IOPRA used here, takes the assortativity as the judgement for whether to do rewiring at each step as long as the directionality remains at $\xi = 1$. The spectral radius λ_1 of the directed networks $G^{(\xi=1)}$ with different assortativities is also shown as a function of the assortativity (circles in Figure 12).

Observations from Figures 10, 11 and 12, all show that the spectral radius λ_1 always decreases with the directionality ξ when the networks have the same degree distribution and the same assortativity ρ_D . Moreover, the degree distribution of the network also influences the change range of the spectral radius λ_1 with ξ . The decrement of the spectral radius λ_1 with ξ increases with the assortativity in binomial directed networks (see Figures 10 and 12 (a)). On the contrary, the decrement of the spectral radius λ_1 with ξ goes down with the assortativity in power-law directed networks (see Figures 11 and 12 (b)). Furthermore, the decrease of the spectral radius in power-law directed networks is larger than that in binomial directed networks, when the assortativity is 0.

Summarizing, the spectral radius λ_1 decreases with the directionality ξ when the assortativity is the same.

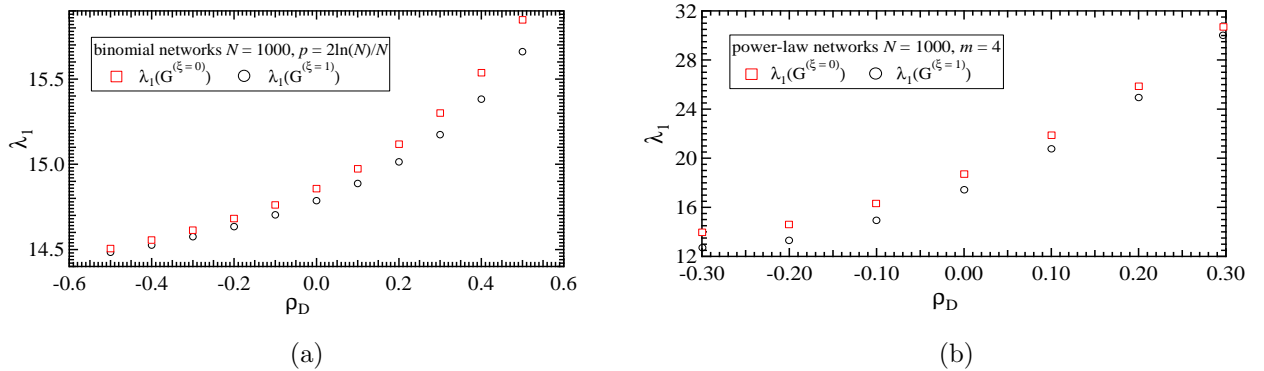


Figure 12: (Color online) The spectral radius as a function of the assortativity (a) in binomial directed networks ($N = 1000, p = 2\ln N/N$) and (b) in power-law directed networks ($N = 1000, m = 4$) for 10^3 network realizations.

In order to protect the network from the virus spreading via increasing the epidemic threshold, while guaranteeing the degree distribution and the assortativity maintained, increasing the directionality of networks is recommended, which, in return enhances the topological robustness.

5 Conclusions

In this paper, we illustrate that many real-world networks are directed. Several algorithms used to generate the directed networks with given directionality ξ are proposed. This allows us to study the influence of the directionality ξ on the spectral properties of networks. The spectral radius λ_1 , which is the inverse of the SIS NIMFA epidemic threshold $\tau_c^{(1)}$, is studied in directed networks. A universal observation is that, the spectral radius decreases with the directionality when the degree distribution and the assortativity of the network is preserved. In order to suppress virus spreading throughout a network, increasing the directionality of the network is advocated to raise the epidemic threshold. The raising range of the epidemic threshold is also influenced by both the degree distribution and the assortativity of the networks. Moreover, the variance of the components of the principal eigenvector decreases with the directionality, and the spectral gap and the algebraic connectivity increase with the directionality, which illustrates that a rise of the directionality enhances the robustness of the networks. We have compared spectral properties in LRA binomial directed networks and IOPRA directed binomial networks. The influence of the difference between in- and out- degree of nodes on spectral properties for power-law directed networks is an open question. The information dissemination is similar to the virus spreading on networks. Our results suggest that encouraging directed online social relations could possibly prevent violent emotions (such as in 2011 London riots) from propagation.

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Appendix

A. Introduction of real-world networks

1. Enron:

This dataset was made public by the Federal Energy Regulatory Commission during its investigations: it is a partially anonymised corpus of e-mail messages exchanged by some Enron employees (mostly part of the senior management). We turned this dataset into a directed graph, whose nodes represent people and with an arc from x to y whenever y was the recipient of (at least) a message sent by x .

2. Ljournal-2008:

LiveJournal is a virtual-community social site started in 1999: nodes are users and there is an arc from x to y if x registered y among his friends. It is not necessary to ask y permission, so the graph is directed). This graph is the snapshot used by Flavio Chierichetti, Ravi Kumar, Silvio Lattanzi, Michael Mitzenmacher, Alessandro Panconesi, and Prabhakar Raghavan in “On compressing social networks”, KDD ’09: Proceedings of the 15th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 219-228, 2009, ACM press, and was kindly provided by the authors.

3. Twitter-2010

Twitter is a website, owned and operated by Twitter Inc., which offers a social networking and microblogging service, enabling its users to send and read messages called tweets. Tweets are text-based posts of up to 140 characters displayed on the user’s profile page.

This is a crawl presented by Haewoon Kwak, Changhyun Lee, Hosung Park, and Sue Moon in “What is Twitter, a Social Network or a News Media?”, Proceedings of the 19th International World Wide Web (WWW) Conference, pages 591-600, 2010, ACM press. Nodes are users and there is an arc from x to y if y is a follower of x . In other words, arcs follow the direction of tweet transmission.

Note that the distance distribution reported in the paper (Figure 4) is quite different from the one we computed (with relative standard error $\leq 0.4\%$ on all points of the cumulative distribution function). Correspondingly, the average distance reported in the paper (4.12) is quite different from our estimate. We have verified that we obtain the distribution reported here even when using breadth-first sampling.

The node numbering of the original graph was not compact, as the node identifiers were Twitter’s actual internal identifiers. We thus renumbered nodes in a contiguous way. The original identifiers can be found in the file with extension .ids. Thus, you can access the Twitter page associated to a node by getting the corresponding identifier and then using the Twitter API. For instance, the node of maximum outdegree is node 2997469, corresponding to identifier 19058681, whose owner can be found at http://api.twitter.com/1/users/show.xml?user_id=19058681.

4. Word Association-2011

The Free Word Association Norms Network is a directed graph describing the results of an experiment of free word association performed by more than 6000 participants in the United States: its nodes correspond to words and arcs represent a cue-target pair (the arc $x \rightarrow y$ means that the word y was output by some of the participants based on the stimulus x).

5. WWW networks

The networks, “cnr-2000”, “in-2004”, “eu-2005”, “uk-2007-05@100000” and “uk-2007-05@1000000” are small WWW networks that were crawled from the Internet. The “cnr-2000” is crawled from the Italian CNR domain. A small crawl of the .in domain performed for the Nagaoka University of Technology is in data “in-2004”. The “eu-2005” is a small crawl of the .eu domain. This network “uk-2007-05@100000” and “uk-2007-05@1000000” have been artificially generated by combining twelve monthly snapshot of the .uk domain and collected for the the DELIS project.

B. Eigenvalues of the directed networks

The spectral radius λ_1 and the spectral gap ($\lambda_1 - \lambda_2$) are considered as important metrics for the percolation processes on networks. Here we also give a whole impression of all eigenvalues in directed networks in a Image-Real figure. We sort the eigenvalues based on the real part of the eigenvalues on 10^3 simulation realizations. If we order them based on the modulus of the eigenvalues, the average of the eigenvalues of 10^3 simulation realizations will be balanced. In that case, the results cannot show how the real and image part of the eigenvalues change with the directionality ξ . The changes of the eigenvalues λ_i with the directionality ξ from 0 to 1 with step 0.1 in binomial directed networks ($N = 10, p = 0.25$) are shown on Figure 13. Surprisingly, the real part of all eigenvalues tends to 0, when the directionality ξ increases.

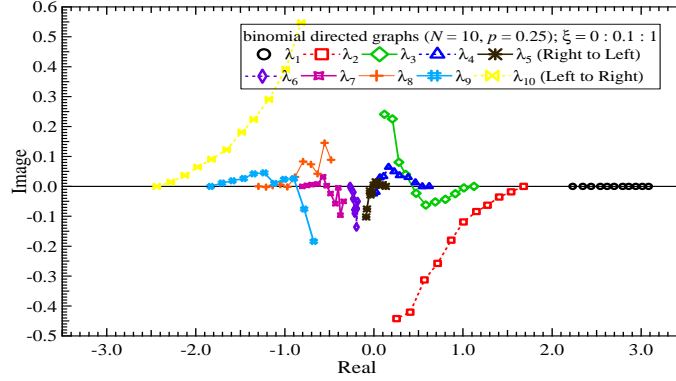


Figure 13: The change of the eigenvalues from bidirected graph to directed graph.

C. Algorithms

Algorithm 1 $IOPRA(G, \xi)$

- 1: Create a bidirectional network $G(N, L)$;
 - 2: Save network $G(N, L)$ as G_s and calculate the directionality ξ_s of network G_s ;
 - 3: **while** $|\xi_s - \xi| > 10^{-5}$ **do**
 - 4: Randomly select two unidirectional links $i \rightarrow j$ and $k \rightarrow l$ associated with the four nodes i, j, k, l ;
 - 5: Rewire the link pair $i \rightarrow j$ and $k \rightarrow l$ into $i \rightarrow l$ and $k \rightarrow j$. The new network G_n is obtained;
 - 6: calculate the directionality ξ_n of the network G_n ;
 - 7: **if** $|\xi_s - \xi| > |\xi_n - \xi|$ **then**
 - 8: $G_s \leftarrow G_n$;
 - 9: $\xi_s \leftarrow \xi_n$;
 - 10: **else**
 - 11: give up this rewired node pair;
 - 12: **end if**
 - 13: **end while**
 - 14: **return** G_s
-

Algorithm 2 $LRA(G, \xi)$

- 1: Create a binomial bidirectional network $G(N, p)$;
 - 2: Randomly choose ξ percentage of bidirectional link pairs;
 - 3: Randomly choose one unidirectional link from each link pair;
 - 4: Randomly reset the chosen unidirectional links to the locations without any link;
 - 5: Save the new network as G_s ;
 - 6: **return** G_s
-